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Applications of Inverse Limits to Extensions of Operators and Approximation of Lipschitz Functions

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We consider extensions of linear operators from finite dimensional subspaces. As a corollary of Steenrod's theorem about inverse limits of topological spaces, we obtain new results concerning approximation in tensor product spaces and the stability of the Cauchy functional equation.
Key Words: extension; approximation; Lipschitz function; inverse limit.

1. INTRODUCTION

Much of analysis deals with the problem of constructing a function or operator with given properties. Probably the best known is the celebrated Hahn–Banach theorem.

Let X be a normed space, let S be a subspace of X, and let $f: S \to \mathbf{R}$ be a linear functional. The Hahn–Banach theorem implies that for every finite dimensional subspace K of X there exists a linear functional $f_K: K \to \mathbf{R}$ such that $f_K|_{S \cap K} = f|_{S \cap K}$ and that $\|f_K\| \leqslant \|f\|$. In this paper, we want to consider the reverse problem of reconstructing f from the family of functionals f_K.

More generally, we want to replace the real numbers $\mathbf{R}$ by a general Banach space V. That is, consider the situation in which we are given a family of operators into V, defined on the finite dimensional subspaces of X, satisfying suitable compatibility conditions. We are going to show that

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if the image space V is a dual space, then by applying the so-called inverse limit we can "glue" operators from finite dimensional subspaces and obtain an operator on the whole space. We also show that the assumption that V is dual is in general essential.

Our results are especially useful in the situation where, having constructed by finite dimensional methods some special operators on finite dimensional subspaces, we want to obtain an operator on the whole space. As an application we generalize some results from [6–8].

2. INVERSE LIMIT AS GLUE

For the convenience of the reader, we first recall from [5] some basic definitions, notation, and results concerning inverse systems of topological spaces.

We say that an ordered set $(\Sigma, \geqslant)$ is *directed* if for every $\sigma, \rho \in \Sigma$ there exists $\tau \in \Sigma$ such that $\tau \geqslant \sigma$ and $\tau \geqslant \rho$.

Suppose that for every σ in a directed set Σ we are given a topological space X_σ and that for every $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$, a continuous mapping $\pi^\sigma_\rho\colon X_\sigma \to X_\rho$ is defined; suppose further that $\pi^\rho_\tau \pi^\sigma_\rho = \pi^\sigma_\tau$ for any σ, $\rho, \tau \in \Sigma$ satisfying $\tau \leqslant \rho \leqslant \sigma$ and that $\pi^\sigma_\sigma = id_{X_\sigma}$ for every $\sigma \in \Sigma$. In this situation we say that the family $\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ is an *inverse system of the spaces* X_σ. An element $\{x_\sigma\}$ of $\prod_{\sigma \in \Sigma} X_\sigma$ is called a *thread* if $\pi^\sigma_\rho x_\sigma = x_\rho$ for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$, and the subspace of $\prod_{\sigma \in \Sigma}$ consisting of all threads is called the *limit of the inverse system* $\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ and is denoted by $\varprojlim X_\sigma$.

STEENROD'S THEOREM [5, Theorem 3.2.13]. *The limit of an inverse system $\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ of nonempty compact spaces is compact and nonempty.*

Any family $\mathscr{T}$ of subsets of a given set can be partially ordered by inclusion. Then $\mathscr{T}$ is directed if for every $T_1, T_2 \in \mathscr{T}$ there exists $T \in \mathscr{T}$ such that $T_1 \cup T_2 \subset T$.

In any case X^T is just the product $\prod_{t \in T} X$; we equip it with the product topology.

The reader will be familiar with the definition of nets (also called generalized sequences), e.g., [4, 5].

Now we establish a general extension theorem.

THEOREM 1. *Let S be a set and let X be a topological space. Let $\mathscr{T}$ be a directed family of subsets of S such that*

$$\bigcup_{T \in \mathscr{T}} T = S.$$

For every $T \in \mathscr{T}$ *let a compact subset* P_T *of* X^T *be given. We assume that*

$$R, T \in \mathscr{T}, T \subset R, f \in P_R \Rightarrow f|_T \in P_T. \tag{1}$$

Then there exists a function $g: S \to X$ *such that*

$$g|_T \in P_T \qquad \textit{for every} \quad T \in \mathscr{T}. \tag{2}$$

Proof. We are going to apply Steenrod's theorem.

For $R, T \in \mathscr{T}, T \subset R$ we define functions $\pi_T^R: P_T \to P_R$ by

$$\pi_T^R(f) = f|_T \qquad \text{for} \quad f \in P_R.$$

From (1) we see that π_T^R is well defined. It is clearly continuous. Thus the family of spaces $\{P_T\}_{T \in \mathscr{T}}$ jointly with the mappings π_T^R forms an inverse system. By the assumptions we know that each P_T is compact, and therefore by Steenrod's theorem the inverse limit $\varprojlim P_T$ is nonempty. This yields the existence of $\{f_T\}_{T \in \mathscr{T}}$ such that

$$f_T \in P_T \qquad \text{for} \quad T \in \mathscr{T}, \tag{3}$$

and

$$(f_R)|_T = f_T \qquad \text{for} \quad R, T \in \mathscr{T}, T \subset R. \tag{4}$$

We define $g := \bigcup_{T \in \mathscr{T}} f_T$. Then (4) yields that g is well defined, while (3) means that g satisfies (2). ∎

3. EXTENSIONS OF OPERATORS

In this section we are going to deal with extensions of operators from finite dimensional subspaces. Our results are connected with the problem of finding minimal projections, whence the notation $\mathscr{P}$ below.

For normed spaces X, V we denote by $\mathscr{L}(X, V)$ the space of all continuous linear operators from X to V. Let S be a subspace of X. For a given operator $A \in \mathscr{L}(S, V)$ we define

$$\mathscr{P}_A(X, V) = \{P \in \mathscr{L}(X, V): P|_S = A\}$$

and

$$\lambda_A(X, V) = \inf\{\|P\|: P \in \mathscr{P}_A(X, V)\}.$$

(This is a mild generalization of similar definitions from [6]; see also [7].) Thus $\mathscr{P}_A(X, V)$ denotes the set of all operator extensions of A and $\lambda_A(X, V)$

the infimum of the norms of all operator extensions of A to all of X. Now we will show that if V is a dual space and $\lambda_A(X,V)<\infty$ then this infimum is obtained. We will need the following lemma, whose standard proof we omit.

LEMMA 2. *Let X be a normed space and let S be a subspace of X. Let V be a Banach space which is a dual space and let $A \in \mathscr{L}(S,V)$.*

For $L \geqslant 0$ we put

$$P_L = \{B \in \mathscr{P}_A(X,V) \colon \|B\| \leqslant L\}.$$

Then P_L is compact in $V^X = \prod_{x\in V} V$, where in V^X we take the product topology with respect to the weak topology in V.*

For the case of projections an analogue of the following result has been proved in [3].

PROPOSITION 3. *Let X be a normed space and let S be a subspace of X. Let V be a Banach space which is a dual space and let $A \in \mathscr{L}(S,V)$.*

If $\lambda_A(X,V)<\infty$ then there exists $B \in \mathscr{P}_A(X,V)$ such that

$$\|B\| = \lambda_A(X,V).$$

Proof. For every $\delta \geqslant 0$ we put

$$P_\delta := \{B \in \mathscr{P}_A(X,V) \colon \|B\| \leqslant \lambda_A(X,V)+\delta\}.$$

By the definition of $\lambda_A(X,V)$, each P_δ is nonempty. By the previous lemma P_δ is compact in V^X. Hence $P_0 = \bigcap_{\delta>0} P_\delta$ is nonempty as an intersection of a decreasing family of nonempty compact sets. Obviously any $B \in P_0$ will do. ∎

Now we are ready to proceed to the main result of the paper. Suppose that we are able to estimate $\lambda_{A|K}(K,V)$ for a certain class $\mathscr{K}$ of subspaces of X, directed with respect to the inclusion order. (For example, $\mathscr{K}$ could be the finite dimensional subspaces.) We show that then $\lambda_A(X,V)$ is the limit of a generalized sequence $\{\lambda_{A|K}(K,V)\}_{K\in\mathscr{K}}$, which partially generalizes Theorem 3.1.6 from [6].

THEOREM 4. *Let X be a normed space and let S be a subspace of X. Let V be a Banach space which is a dual space and let $A \in \mathscr{L}(S,V)$.*

Let $\mathscr{K}$ be a directed family of subspaces of X such that $S \subset \bigcup_{K\in\mathscr{K}} K$ and such that $\bigcup_{K\in\mathscr{K}} K$ is dense in X. Then

$$\lambda_A(X,V) = \lim_{K\in\mathscr{K}} \lambda_{A|_{S\cap K}}(K,V). \tag{5}$$

Proof. For any subspaces X_1, X_2 of X with $X_1 \subset X_2$, it is easy to see that

$$\lambda_{A|_{S \cap X_1}}(X_1, V) \leqslant \lambda_{A|_{S \cap X_2}}(X_2, V).$$

Thus to prove (5) it will be enough to show that

$$\lambda_A(X, V) \leqslant \sup_{K \in \mathscr{K}} \lambda_{A|_{S \cap K}}(K, V). \tag{6}$$

We are going to apply Theorem 1. We take in V weak* topology. It is enough to consider the case

$$L := \sup_{K \in \mathscr{K}} \lambda_{A|_{S \cap K}}(K, V) < \infty.$$

For every $K \in \mathscr{K}$ we put

$$P_K := \{B \in \mathscr{P}_{A|_{S \cap K}}(K, V) \colon \|B\| \leqslant L\}.$$

Clearly by Lemma 2 and Proposition 3, P_K is compact and nonempty in V^K for every $K \in \mathscr{K}$ (the topology in V^K is the product topology with respect to the weak* topology in V). By Theorem 1 we obtain a function $C \colon \bigcup_{K \in \mathscr{K}} K \to V$ such that

$$C|_K \in P_K \qquad \text{for every} \quad K \in \mathscr{K}. \tag{7}$$

This implies that $C|_K$ is an operator such that $\|C|_K\| \leqslant L$ for every $K \in \mathscr{K}$, and consequently, that C is a linear operator with $\|C\| \leqslant L$. By definition of P_K, (7) implies also that $C|_{S \cap K} = A|_{S \cap K}$ for every $K \in \mathscr{K}$. By assumption, $S \subset \bigcup_{K \in \mathscr{K}} K$, which yields that $C|_S = A$. As C is a linear bounded operator defined on a dense subset of X, it has a unique continuous extension D to all of X. Then clearly $D|_S = A$ and $\|D\| \leqslant L$. This means that $\lambda_A(X, V) \leqslant L$, which proves (6). ∎

COROLLARY 5. *Let X be a normed space and let S be a subspace of X. Let V be a finite dimensional Banach space, and let $A \in \mathscr{L}(S, V)$.*

Let $\mathscr{K}$ be the family of all finite dimensional subspaces of X. Then

$$\lambda_A(X, V) = \lim_{K \in \mathscr{K}} \lambda_{A|_{S \cap K}}(K, V).$$

EXAMPLE 6. Now we will show that the assumption that V is a dual space in Theorem 4 cannot be dispensed with.

As usual we denote by ℓ^∞ the Banach space of all real bounded sequences with the supremum norm, by c_0 the subspace of ℓ^∞ consisting of all sequences with limit zero, and by $c_{00} \subset c_0$ the space of all sequences which are eventually zero.

Let $\mathscr{K}$ denote the set of all finite dimensional subspaces of ℓ^∞. Let us first notice that $\lambda_{id|_{c_{00} \cap K}}(K, c_0) = 1$ for every $K \in \mathscr{K}$. This follows from the fact that for every $K \in \mathscr{K}$ there exists $n \in \mathbf{N}$ such that for every $x \in c_{00} \cap K$ we have $x_k = 0$ for every $k > n$. Thus we can define a norm one extension P of the identity operator from $c_{00} \cap K$ onto K by

$$P(x_1, \ldots, x_n, x_{n+1}, \ldots) = (x_1, \ldots, x_n, 0, 0, \ldots).$$

However, $\lambda_{id|c_{00}}(\ell^\infty, c_0) = \infty$, as it is well known there is no bounded linear projection from ℓ^∞ onto c_0. See for example [1] for various proofs of this and some historical discussion.

Remark 7. The Hahn–Banach theorem "works" by prolonging the functional by transfinite induction or by the Kuratowski–Zorn lemma. We would like to mention that the technique used here (i.e., the existence of inverse limit of compact spaces) can also provide a proof.

So let X be a normed space and let S be a subspace of X. Let $f: S \to \mathbf{R}$ be a linear functional. By the finite dimensional case of the Hahn–Banach theorem, we have $\lambda_{f|_{S \cap K}}(K, \mathbf{R}) \leqslant \|f\|$ for every finite dimensional subspace K of X. By Corollary 5 this implies that $\lambda_f(S, \mathbf{R}) = \|f\|$. Hence by Proposition 3 there exists $g \in \mathscr{L}(X, \mathbf{R})$ such that $\|g\| = \|f\|$ and $g|_S = f$.

4. EXTENSIONS IN TENSOR PRODUCT SPACES

For some basic results concerning tensor products we refer the reader to [2] (see also [7]). We only mention that a norm α on $X \times Y$ is called a *crossnorm* if $\alpha(x \otimes y) = \|x\| \cdot \|y\|$ for all $x \in X$, $Y \in Y$. Given a norm α on $X \otimes Y$, one denotes by $X \otimes_\alpha Y$ the completion of $X \otimes Y$ with respect to this norm. Given spaces X, Y with norm α on $X \otimes Y$ and operators $A \in \mathscr{L}(X, X)$, $B \in \mathscr{L}(Y, Y)$, we denote by $A \otimes B$ the tensor product of A and B. If $A \otimes B \in \mathscr{L}(X \otimes Y, X \otimes Y)$ then by $A \otimes_\alpha B \in \mathscr{L}(X \otimes \alpha Y, X \otimes_\alpha Y)$ we denote the unique continuous extension of $A \otimes B$ to $X \otimes_\alpha Y$. Clearly we have $\|A \otimes B\| = \|A \otimes_\alpha B\|$.

In this section we are going to correct and generalize results from [7]. First we show with the following example that the proof of the main theorem of that paper, namely Theorem 2.5, contains a gap.

EXAMPLE 8. Let X be a separable Banach space and let X_n be an increasing sequence of finite dimensional subspaces of X such that $X = \overline{\bigcup_n X_n}$. Let V be a subspace of X_1 and let S be a subspace of X which contains V. Put $S_n = X_n \cap S$. It is claimed in the proof of [7, Theorem 2.5] that for every $M \in \mathscr{L}(S, V)$ we have

$$\lambda_M(X, V) = \lim_{n \to \infty} \lambda_{M|_{S_n}}(X_n, V).$$

We show that this need not be the case. Let X and X_n be chosen as above. Let $e \in X_1$ and $f \in X \setminus \bigcup_n X_n$ be chosen arbitrarily. We put $V = \mathbf{R} \cdot e$, $S = \mathbf{R} \cdot e + \mathbf{R} \cdot f$. Clearly $V \subset S \subset X$, $V \subset X_1$. Let $M \in \mathscr{L}(S, V)$ be defined by

$$M(\alpha e + \beta f) = \beta e.$$

Then $M|_{S_n} = 0$, so $\lim_{n \to \infty} \lambda_{M|_{S_n}}(X_n, V) = 0$. However, by the definition of M we have $\lambda_M(X, V) > 0$.

There are several ways that the gap contained in the proof of Theorem 2.5 from [7] can be removed. Perhaps the most natural is simply to assume that the sequence of finite dimensional subspaces $S_n = S \cap X_n$ considered in the proof satisfies the following additional condition (c):

$$\overline{\bigcup_n S_n} = S.$$

This can easily be arranged since S (being a subspace of the separable Banach space X) is also separable. One has only to choose the subspaces S_n first, and then choose suitable X_n. Of course the sequence S_n constructed in Example 8 does not satisfy (c).

For the convenience of the reader we state Theorem 2.5 from [7]. The special case when X and Y are finite dimensional, which is correctly proved in [7], will be needed in the proof of Theorem 10.

THEOREM L. *Let X, Y be separable Banach spaces (complex or real). Suppose $V \subset X$ and $W \subset Y$ are finite dimensional linear subspaces. Let $V \subset S$ and $W \subset Z$, where S is a subspace of X and Z is a subspace of Y, and let $M \in \mathscr{L}(S, V)$, $N \in \mathscr{L}(Z, W)$ be given. If α is a reasonable crossnorm on $X \otimes Y$ then*

$$\lambda_{M \otimes N}(X \otimes_\alpha Y, V \otimes_\alpha W) \geqslant \lambda_M(X, V)\, \lambda_N(Y, W).$$

We will need the following simple lemma.

LEMMA 9. *Let X, Y be Banach spaces. Suppose that $V \subset X$ and $W \subset Y$ are finite dimensional subspaces. Let S be a subspace of X and Z be a* *subspace of* Y*, and let* $M \in \mathscr{L}(S,V)$*,* $N \in \mathscr{L}(Z,V)$ *be given. If* $M \otimes N \in \mathscr{L}(S \otimes Z, X \otimes Y)$ *then*

$$\lambda_{M \otimes N}(X \otimes Y, V \otimes W) = \lambda_{M \otimes_\alpha N}(X \otimes_\alpha Y, V \otimes_\alpha W).$$

Proof. Let $F_\alpha \in \mathscr{P}_{M \otimes_\alpha N}(X \otimes_\alpha Y, V \otimes_\alpha W)$ be arbitrary. Then clearly $F = F_\alpha|_{S \otimes Z} \in \mathscr{P}_{M \otimes N}(X \otimes Y, V \otimes W)$. Since $X \otimes Y$ is dense in $X \otimes_\alpha Y$ we obtain the equality $\|F\| = \|F_\alpha\|$ which yields

$$\lambda_{M \otimes N}(X \otimes Y, V \otimes W) \leqslant \lambda_{M \otimes_\alpha N}(X \otimes_\alpha Y, V \otimes_\alpha W).$$

Now we establish the reverse inequality. Let $F \in \mathscr{P}_{M \otimes N}(X \otimes Y, V \otimes W)$ be arbitrary. Then F has a unique continuous extension F_α to $X \otimes_\alpha Y$ and of course $\|F_\alpha\| = \|F\|$. Now

$$F_\alpha|_{S \otimes Z} = M \otimes N.$$

Moreover, since $S \otimes Z$ is dense in $S \otimes_\alpha Z$ we obtain that

$$F_\alpha|_{S \otimes_\alpha Z} = M \otimes_\alpha N,$$

which implies that $F_\alpha \in \mathscr{P}_{M \otimes_\alpha N}(X \otimes_\alpha Y, V \otimes_\alpha W)$. ∎

Applying Theorems L and 4 we are able to correct and generalize Theorem 2.5 from [7]. In particular, we show that the assumption that X and Y be separable is not essential.

THEOREM 10. *Let* X*,* Y *be Banach spaces. Suppose that* $S \subset X$ *and* $Z \subset Y$ *are subspaces. Let* V *be a finite dimensional subspace of* S *and* W *be a finite dimensional subspace of* Z*, and let* $M \in \mathscr{L}(S,V)$*,* $N \in \mathscr{L}(Z,W)$ *be given. If* α *is a reasonable crossnorm on* $X \otimes Y$ *with respect to which* $M \otimes N \in \mathscr{L}(S \otimes Z, V \otimes W)$ *then*

$$\lambda_{M \otimes N}(X \otimes_\alpha Y, V \otimes_\alpha W) \geqslant \lambda_M(X,V)\, \lambda_N(Y,W).$$

Proof. Let $\mathscr{K}_X$ and $\mathscr{K}_Y$ denote the class of all finite dimensional subspaces of X and Y, respectively. Clearly $S \subset X = \bigcup_{K_X \in \mathscr{K}_X} K_X$ and $W \subset Y = \bigcup_{K_Y \in \mathscr{K}_Y} K_Y$, so by Corollary 5 we obtain that

$$\lambda_M(X,V) = \lim_{K \in \mathscr{K}_X} \lambda_{M|_{S \cap K}}(K,V), \tag{8}$$

$$\lambda_N(Y,W) = \lim_{K \in \mathscr{K}_Y} \lambda_{N|_{Z \cap K}}(K,W). \tag{9}$$

Now let

$$\mathscr{K} := \{K_X \otimes K_Y : K_X \in \mathscr{K}_X,\ K_Y \in \mathscr{K}_Y\}.$$

Then $\bigcup_{K \in \mathscr{K}} K$ equals $X \otimes Y$. Applying Corollary 5 once more we obtain that

$$\lambda_{M \otimes N}(X \otimes Y, V \otimes W) = \lim_{K_X \otimes K_Y \in \mathscr{K}} \lambda_{(M \otimes N)|(S \cap K_X) \otimes (W \cap K_Y)}(K_X \otimes K_Y, V \otimes W).$$

By (8), (9), and the finite dimensional case of Theorem L, we now see that

$$\lambda_{M \otimes N}(X \otimes Y, V \otimes W) \geqslant \lambda_M(X, V)\, \lambda_N(Y, W).$$

Lemma 9 makes the proof complete. ∎

As a corollary we see that the assumptions in Theorems 2.5 and 2.6 from [7] about X and V being separable are both superfluous. (The proof presented in [7] holds, except that in the last part of the proof of Theorem 2.5 one applies our Corollary 5 instead of the reasoning from [6].) This yields in particular that the assumption in Theorem 3.1 from [7], that S, T are metrizable or separable, is superfluous.

5. CAUCHY FUNCTIONAL EQUATION

In this section we show that Theorem 1 yields new results about Lipschitz stability of the Cauchy functional equation. Let X, V be normed spaces and let $D \subset X$. By the Cauchy difference of a given function $f: D \to V$ we mean the function $\mathscr{C}f: C(D) \to V$ defined by

$$\mathscr{C}f(x, y) := f(x+y) - f(x) - f(y) \qquad \text{for} \quad (x, y) \in C(D),$$

where $C(D) = \{(x, y) \in X \times X : x, y, x+y \in D\}$. One notices easily that a function $f: X \to V$ is additive iff its Cauchy difference is identically zero. Thus, in a certain sense, the Cauchy difference measures the "distance" of the given function from the space of additive functions.

If $0 \in D$, then we denote by $\mathrm{lip}(D, V)$ the space of all Lipschitz functions from D to V with the norm

$$\|f\|_{\mathrm{lip}} := \|f(0)\| + \mathrm{lip}(f),$$

where $\mathrm{lip}(f)$ denotes the smallest Lipschitz constant of f. For the reader's convenience we quote the main result from [8] dealing with the local Lipschitz stability of the Cauchy functional equation.

THEOREM TT. *Let X, V be finite dimensional normed spaces, let $D \subset X$ be a convex set such that $0 \in D$, and let $f: D \to V$ be a function such that*

$\mathscr{C}f \in \mathrm{lip}(C(D), V)$. Then there exists an additive function $a: X \to V$ such that $f - a|_D \in \mathrm{lip}(D, V)$ and

$$\|f - a|_D\|_{\mathrm{lip}} \leqslant \|\mathscr{C}f\|_{\mathrm{lip}}.$$

We would like to mention that the main tool in the proof of the above theorem is Rademacher's theorem. Thus the proof has strictly finite dimensional character. We will show that Theorem 1 leads to the same conclusion, without the assumption of the finite dimensionality of X.

THEOREM 11. *Let X be a normed space and let V be a finite dimensional Banach space. Let $D \subset X$ be a convex set such that $0 \in D$, and let $f: D \to V$ be a function such that $\mathscr{C}f \in \mathrm{lip}(C(D), V)$. Then there exists an additive function $A: X \to V$ such that $f - A \in \mathrm{lip}(D, V)$ and*

$$\|f - A|_D\|_{\mathrm{lip}} \leqslant \|\mathscr{C}f\|_{\mathrm{lip}}.$$

Proof. Without loss of generality we may assume that D generates X.

Let K be an arbitrary finite dimensional subspace of X. It is easily seen that $\mathrm{int}_K(D \cap K) \neq \varnothing$. For if $d_1, ..., d_n \in D$ form a linearly independent set which generates K, then $\frac{1}{n+1} d_1 + \cdots \frac{1}{n+1} d_n \in \mathrm{int}_K(D \cap K)$, since $0 \in D \cap K$.

Let

$$P_K := \{a: K \to V \text{ additive} \mid \|f|_{D \cap K} - a|_{D \cap K}\|_{\mathrm{lip}} \leqslant \|\mathscr{C}f\|_{\mathrm{lip}}\}.$$

We are going to show that the assumptions of Theorem 1 are satisfied. We take in V the norm topology and in V^X take the product topology.

Let us first notice that $\|\mathscr{C}(f|_K)\|_{\mathrm{lip}} \leqslant \|\mathscr{C}f\|_{\mathrm{lip}}$, so by Theorem TT we obtain that P_K is nonempty for each finite-dimensional subspace K of X. We will now show that P_K is compact in V^K. Directly from the definition one can check that P_K is closed. It is now enough to show that P_K is contained in a compact set. Let $a_0 \in P_K$ be arbitrarily chosen and let

$$W(K) := \{g \in V^k \mid \|g(k)\| \leqslant 2\,\|\mathscr{C}f\|_{\mathrm{lip}}\,\|k\| \text{ for } k \in K\}.$$

As V is finite dimensional the set $\{y \in V: \|y\| \leqslant 2\,\|\mathscr{C}f\|_{\mathrm{lip}}\,\|k\|\}$ is compact in V for every $k \in K$, so $W(K)$ is compact as a product of compact sets. This implies that $a_0 + W(K)$ is also compact. We show that $P_K \subset a_0 + W(K)$. Let $a \in P_K$ be arbitrary. Then

$$\|a|_{D \cap K} - a_0|_{D \cap K}\|_{\mathrm{lip}} \leqslant \|f|_{D \cap K} - a|_{D \cap K}\|_{\mathrm{lip}} + \|f|_{D \cap K} - a_0|_{D \cap K}\|_{\mathrm{lip}} \leqslant 2\,\|Cf\|_{\mathrm{lip}},$$

which means that $a - a_0$ is an additive function which is Lipschitz with constant $2\,\|\mathscr{C}f\|_{\mathrm{lip}}$ on the set $D \cap K$ which has nonempty interior in K. This yields that $\|a - a_0\|_{\mathrm{lip}} \leqslant 2\,\|\mathscr{C}f\|_{\mathrm{lip}}$ and consequently that $a - a_0 \in W(K)$.

Thus we have obtained that for every finite dimensional subspace K of X the set P_K is nonempty and compact. By Theorem 1 we obtain that there exists a function $A\colon X \to V$ such that $A|_K \in P_K$ for every finite dimensional subspace K of X. However, this yields trivially that A is additive and that the conclusion of the theorem holds. ∎

Since we have shown that the assumption in Theorem TT that X is finite dimensional is not essential, it seems natural to ask if the same is true about V.

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